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ABS ALGORITHMS FOR SOLVING LINEAR INEQUALITIES①

Zhao Jin-xi (赵金熙)

Abstract *We consider a class of ABS type algorithms for solving system of linear inequalities, where the number of inequalities does not exceed the number of variables.*

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1. Introduction

Consider solving the linear system of inequalities

$$a_i^T x \leqslant b_i \qquad i \in I = \{1,2,\cdots,m\}, \tag{1}$$

where

$$a_i \in R^n,\ a_i \neq 0,\ x \in R^n,\ b_i \in R,\ m \leqslant n.$$

Let A_m be an $n \times m$ matrix

$$A_m = [a_1, a_2, \cdots, a_m]$$

and B_m be a vector in R^m with

$$B_m = (b_1, b_2, \cdots, b_m)^T.$$

Then we can express problem (1) as

$$A_m^T x \leqslant B_m. \tag{2}$$

For each $i \in I$ define the closed half space

$$L_i \equiv \{x \in R^n \mid a_i^T x \leqslant b_i\}$$

and its boundary hyperplane

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$$S_i \equiv \{x \in R^n | a_i^T x = b_i\}.$$

Define $L \equiv \bigcap_{i\in I} L_i$ and assume throughout that problem (2) is always consistent. The ABS algorithms (see Abaffy, Broyden, Spedicato [1]) are a class of algorithms for solving the linear system

$$A_m^T x = B_m.$$

The ABS class consists of the following steps:

(A) Let $x_1 \in R^n$ be arbitrary, $H_1 \in R^{n,n}$ be arbitrary nonsingular and set $i=1$.

(B) Compute the search vector p_i by

$$p_i = H_i^T z_i,$$

where $z_i \in R^n$ is arbitrary save that

$$z_i^T H_i a_i \neq 0.$$

(C) Update the approximation of the solution by

$$x_{i+1} = x_i - \alpha_i p_i,$$

and

$$\alpha_i = (a_i^T x_i - b_i) / p_i^T a_i.$$

If $i=m$ then stop, $x_{m+1} \in S \equiv \bigcap_{i\in I} S_i$, i.e., x_{m+1} solves the system, otherwise

(D) Update the matrix H_i by

$$H_{i+1} = H_i - H_i a_i w_i^T H_i,$$

where w_i is an arbitrary element in R^n satisfying

$$w_i^T H_i a_i \neq 0$$

and return to (B).

The problem of solving system of linear inequalities arises in numerous fields, e.g., in linear programming, or in image reconstruction from projections [6]. In this paper, we generalize the ABS algorithms for solving linear inequalities (2) and we discuss some properties of the obtained algorithms. Numerical results show the efficiency of the given methods.

2. The Methods of the ABS Class for Linear Inequalities

Assume that the matrix A_m in problem (2) is a full rank matrix. The ABS algorithm for sloving linear inequalities is based on the following steps:

The ABS algorithm for linear inequalities.

Let H_1 be an arbitrary nonsingular matrix in $R^{n,n}$, x_1 be an arbitrary vector in R^n. For $i=1,2,\cdots,m$, compute

(A) $p_i = H_i^T z_i$,

where $z_i \in R^n$ is an arbitrary vector satisying

$$a_i^T H_i^T z_i = a_i^T p_i > 0.$$

(B) $x_{i+1} = x_i + t_i p_i$,

where

$$t_i \leqslant (b_i - a_i^T x_i) / a_i^T p_i. \tag{3}$$

(C) $H_{i+1} = H_i - H_i a_i w_i^T H_i$,

where $w_i \in R^n$ is an arbitrary vector satisfying

$$w_i^T H_i a_i = 1. \tag{4}$$

Then $X_{m+1} \in L$, i.e., x_{m+1} is a solution of the problem (2). This algorithm retains all properties of ABS methods which do not depend on the vector B_m, for instance,

(a) the matrix $A_k^T P_k$ is nonsingular lower triangular, where

$$A_k = [a_1, a_2, \cdots, a_k], \qquad P_k = [p_1, p_2, \cdots, p_k].$$

(b) $H_k A_{k-1} = 0, \quad H_k^T W_{k-1} = 0, \quad W_{k-1} = [w_1, w_2, \cdots, w_{k-1}]$.

We are particularly interested in the symmetric ABS class, which contains as a special algorithm the Huang algorithm, defined by the following choices of the available parameters

$$H_1 = I, \quad x_1 = 0, \quad w_k = a_k / a_k^T H_k a_k, \quad z_k = a_k, \tag{5}$$

corresponding to the following algorithm, since the given choices imply $a_k^T p_k > 0$.

Huang method for linear inequalities.

Let $H_1 = I$, $x_1 = 0$.

For $k = 1,2,\cdots,m$, calculate

(a) $p_k = H_k a_k$,

(b) $x_{k+1} = x_k + t_k p_k$,

where

$$t_k \leqslant (b_k - a_k^T x_k) / p_k^T a_k. \tag{6}$$

(c) $H_{k+1} = H_k - p_k p_k^T / p_k^T a_k$.

Then $x_{m+1} \in L$, i.e., x_{m+1} is a solution of the linear inequalities (2).

A numerically more stable version of (5) is the modified Huang method[2] defined by

$$H_1 = I, \quad z_k = H_k a_k, \quad w_k = z_k / z_k^T z_k.$$

The following properties of the solution vector generated by the ABS class for linear equations are known from Abaffy and Spedicato [2].

Let

$$x^* = x_{i+1} + H_{i+1}^T y$$

and $y \in R^n$ arbitrary.

Then

(a) $x^* \in \bigcap_{i=1}^{i} S_i$, i.e., x^* is a solution vector of the first i equations of problem

$$A_m^T x = B_m.$$

(b) Let

$$X = \{x | x = x_{i+1} + H_{i+1}^T y, \ y \in R^n\}.$$

Then

$$X = \bigcap_{i=1}^{i} S_i.$$

Moreover, if the solution vectors $\{x_i\}$ are generated by the Huang method, and

$$x_1 = \beta \alpha_1,$$

$\beta \in R$ arbitrary, then

(c)

$$\|x_i\| \leqslant \|x_{i+1}\|,$$
$$\|x_{i+1} - x\|_{x \in S} \leqslant \|x_i - x\|_{x \in S}$$

for $i = 2, \cdots, m$ and x_{i+1} is the solution of least euclidean norm of the first i equations.

For linear inequalities we can obtain the following theorem.

Theorem 1 *If the vectors $a_1, a_2, \cdots, a_k$ are linearly independent, then the vectors x_{k+1}, $k = 1, 2, \cdots, m$ computed by ABS algorithm for linear inequalities possess the following properties:*

(a) x_{k+1} *is a solution of the linear inequalities*

$$A_k^T x \leqslant B_k, \quad k = 1, 2, \cdots, m,$$

i.e.,

$$x_{k+1} \in \bigcap_{i=1}^{k} L_i.$$

(b) *Let* $y \in R^n$ *be an arbitrary vector, then the vector*

$$x = x_{k+1} + H_{k+1} y \tag{7}$$

is also a solution of the problem (2), *i.e.,*

$$x \in \bigcap_{i=1}^{k} L_i.$$

Proof The proof of this theorem is completed by induction.

When $k=2$,

$$x_2 = x_1 + t_1 p_1,$$

and it is immediate to verify that $x_2 \in L_1$ for t_1 given by (3).

Suppose that conclusion (a) is valid until the $(k-1)$th step, i.e.,

$$x_k \in \bigcap_{i=1}^{k-1} L_i. \tag{8}$$

We need to prove that

$$x_{k+1} \in \bigcap_{i=1}^{k} L_i.$$

It is seen immediately that $x_{k+1} \in L_k$ from (3), while

$$x_{k+1} \in \bigcap_{i=1}^{k-1} L_i$$

from the induction assumption and property (a). Hence we have

$$x_{k+1} = x_k \in \bigcap_{i=1}^{k} L_i.$$

Property (b) also follows easily from (3).

3. Pivoting Strategies

In the implementation of the Huang method for linear inequalities selecting the pivot is important for numerical stability and the determination of the numerical rank of A_m. In the following, we give a possible strategy. Given the starting vector $x_1 = 0$, $H_1 = I$, define $p_i^{(j)}$ by

$$p_i^{(j)} = H_i a_j \quad , \qquad j = i, i+1, \cdots, m.$$

We can select an index s such that

$$s = \arg[\max_{i \leqslant s \leqslant m} (a_s^T p_i^{(s)})].$$

We choose at the i–th step of the Huang algorithm the s–th inequality to deal with, which implies that we exchange the s–th column and i–th column of matrix A_m, the s–th element and i–th element of vector B_m, respectively. We also exchange position of $p_i^{(i)}$ and $p_i^{(s)}$. Then let

$$p_i = p_i^{(i)}.$$

If

$$\|p_{r+1}\|_2 \leqslant \varepsilon \|p_1\|_2,$$

at the $(r+1)$th step, then we assume that the numerical rank of the matrix A_{r+1} is equal to the numerical rank of A_r. Using Huang method, we have

$$H_{i-1} = I - \sum_{k=1}^{i-1} (p_k p_k^T / p_k^T a_k). \tag{9}$$

By selecting the pivot and by proper exchange of columns, we obtain the following recurrence relations

$$p_i^{(j)} = p_{i-1}^{(j)} - [(p_{i-1}^{(i)})^T p_{i-1}^{(j)} / p_{i-1}^T a_{i-1}] p_{i-1}. \tag{10}$$
$$j = i, i+1, \cdots, m.$$

If $\text{rank}(A_{r+1}) = \text{rank}(A_r)$, then we do not consider the $(r+1)$th inequality and we get

$$x_{r+1} = \sum_{i=1}^{r} t_i p_i. \tag{11}$$

Thus

$$x_{r+1} \in \bigcap_{i=1}^{r} L_i.$$

In general, x_{r+1} is not always a solution of problem (2), i.e.,

$$x_{r+1} \notin \bigcap_{i=1}^{m} L_i.$$

The following theorem 2 was given in [4].

Theorem 2 *Let $a_1, a_2, \cdots, a_k, a_{k+1}$ be a vector system, and $b_1, b_2, \cdots, b_{k+1}$ be real numbers. If the linear system*

$$\begin{cases} a_i^T x = b_i \\ a_{k+1}^T x \leqslant b_{k+1}, \end{cases} \tag{12}$$
$$1 \leqslant i \leqslant k,$$

is consistent, then every solution vector of

$$a_i^T x = b_i \qquad 1 \leqslant i \leqslant k \tag{13}$$

satisfies (12) *if and only if* a_{k+1} *is a linear combination of the vector system* $a_1, a_2, \cdots, a_k$.

We assume $\operatorname{rank}(A_m) = k$ and that $a_1, a_2, \cdots, a_k$ is a maximal set of linear independent vectors. With this assumption and using the properties of the ABS algorithm Theorem 2 can be generalized as follows.

Theorem 3 *If the linear system*

$$\begin{cases} a_i^T x = b_i, \\ a_j^T x \leqslant b_j, \end{cases} \tag{14}$$

$$i = 1,2,\cdots,k; \qquad j = k+1, k+2, \cdots, m$$

is consistent, then the vector x_{k+1} *obtained by the k–th step of the ABS algorithms for linear equations is also a solution vector of the problem* (14). *Moreover, with arbitrary* $y \in R^n$

$$x = x_{k+1} + H_{k+1} y,$$

is also a solution of the problem (14).

Theorem 3 makes the possibility of solving linear overdetermined consistent inequalities (i.e., $m \geqslant n$) by the ABS algorithms. This will be discussed in another paper.

Remark From the proof of Theorem 2 it follows that condition

$$b_{k+1} \geqslant \sum_{i=1}^{k} \alpha_i b_i$$

is necessary for the consistency of the system under the given dependency condition on α_{k+1}.

4. Numerical Experiments

The algorithm described (modified Huang method) above has been implemented with the choice

$$t_k = (b_k - a_k^T x_k) / p_k^T a_k - ep, \qquad ep = 0.3,$$

and has been run on CISCA Vax–6320 computer in double precision (about 15 significant digits). The considered problems have the following data.

Problem 1.

$$A = [a_{ij}] = [1/(i+j-1)] \in R^{n,n}, \qquad n = 15$$

$$B = (b_1, b_2, \cdots, b_{15})^T, \qquad b_i = \sum_{j=1}^{n} a_{ij},$$

$$i = 1,2,\cdots,15,$$

starting vector $x_1 = (1,8,\cdots,3375)^T$.

Problem 2

$$A = [a_{ij}] = [\max(i,j)] \in R^{n,n}, \quad n = 30$$

$$B = (b_1, b_2, \cdots, b_{30})^T, \quad b_i = \sum_{j=1}^{n} a_{ij},$$

$$i = 1,2,\cdots,30,$$

starting vector $x_1 = (1,8,\cdots,27000)$.

Table 1. Solution vector of the problem 1.

−.12323E+02	+.23742E+03	−.14037E+04	+.36440E+04	−.41165E+04
+.14908E+04	−.20301E+03	+.31986E+03	+.13512E+04	−.21419E+04
+.31545E+04	−.31264E+04	−.23793E+04	+.51838E+04	−.19831E+04

Table 2. Residual vector R = B−Ax of the problem 1.

.47413E+00	.84695E−02	.76979E−04	.51705E−06	.27208E−08
.11446E−10	.72830E−13	.26645E−13	.15987E−13	.24868E−13
.14218E−13	.19539E−13	.88817E−14	.14210E−13	.35527E−14

Table 3. Solution vector of the problem 2.

.28372E04	−.28351E−04	.10000E01	.10006E01	.10000E01
.10000E01	.10000E01	.10000E01	.10000E01	.10000E01
.10000E01	.10000E01	.10001E01	.10000E01	.10001E01
.10001E01	.10001E01	.10002E01	.10002E01	.10003E01
.10004E01	.10006E01	.10008E01	.10012E01	.10018E01
.10029E01	.10051E01	.10103E01	.10258E01	.94306E00

Table 4. Residual vector R = B−Ax of the problem 2.

.28364E04	.29996E00	.29987E00	.29971E00	.29949E00
.29920E00	.29855E00	.29843E00	.29793E00	.29737E00
.29672E00	.29599E00	.29517E00	.29424E00	.29319E00
.29200E00	.29065E00	.28910E00	.28732E00	.28525E00
.28282E00	.27990E00	.27636E00	.27196E00	.26632E00
.55883E00	.24839E00	.23279E00	.20685E00	.15508E00

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Zhao Jin-xi Department of Mathematics, Nanjing University, Nanjing 210008, PRC.